Journal of Statistical Physics, Vol. 38, Nos. 3/4, 1985

Time Evolution of a One-Dimensional Point System: A Note on Fritz's Paper

E. Presutti[1] **and E. Scacciatelli**[1]

Received October 9, 1984

In a companion paper (Ref. 5) Fritz studies the time evolution of a one-dimensional point system which was introduced by Spitzer as a model of traffic. In the present paper we improve some of the results of Ref. 5 by using a different approach. Our results are obtained in a very simple and straightforward way but the techniques employed require conditions somewhat stronger than those assumed in Ref. 5.

KEY WORDS: Nonlinear diffusion equations; hydrodynamical behavior; coupling of random walks.

There is a very simple and direct way to derive and improve some of the results of Ref. 5, as we shall see in this paper. Our method, as compared to that of Ref. 5, requires stronger assumptions and our techniques seem inadequate for studying extensions to many-dimensional cases. On the other hand our procedure looks very simple and more flexible so it might become useful in the analysis of other models.

We consider the equations

$$\frac{d}{dt}\delta(n,t) = U'(\delta(n+1,t)) + U'(\delta(n-1,t)) - 2U'(\delta(n,t)) \tag{1a}$$

$$\delta(n,0) = \delta(n) \tag{1b}$$

where $n \in \mathbb{Z}$, $t \geqslant 0$, $U \in C^\infty(\mathbb{R})$, and it is a convex symmetric function of $r \in \mathbb{R}$. Throughout we shall consider the case when there are positive constants a, b, δ', δ'' so that

$$0 < a \leqslant U''(r) \leqslant B < \infty \tag{2}$$

$$0 < \delta' \leqslant \delta(n) \leqslant \delta'' < \infty \tag{3}$$

[1] Dipartimento Matematico, Università di Roma, Rome, Italy.

We refer to Ref. 5 for a discussion on the interpretation and meaning of Eq. (1). It is easy to see that Eq. (1) has a unique solution and that $\delta(n, t) = \delta$, for all $n \in \mathbb{Z}$ and all $t \geqslant 0$, is a solution of Eq. (1a), i.e., $\delta(n) = \delta$ is a stationary profile. It is less obvious but still true that these are the only stationary profiles; see the Remarks to Lemma 2 below.

The question we are interested in concerns the time evolution of "slowly varying" initial profiles, i.e., the hydrodynamical behavior of Eq. (1); cf. Ref. 5 for the use of such terminology and Ref. 3 for a survey on the subject. We state precisely what is the problem and its answer in the following.

Theorem 1. Assume $U \in C^\infty(\mathbb{R})$ is a symmetric and convex function and that Eq. (2) holds. Let $\varepsilon \in (0, 1]$ and denote by $\delta_\varepsilon(n, t)$ the solution of Eq. (1) with initial datum $\delta_\varepsilon(n, 0) = \Delta(\varepsilon n)$, where $\Delta(r)$ is in $C^\infty(\mathbb{R})$ and

$$0 < \delta' \leqslant \Delta(r) \leqslant \delta'' < \infty, \qquad \|\Delta'\| = \sup_{r \in \mathbb{R}} |\Delta'(r)| \leqslant \delta''' < \infty \tag{4}$$

for suitable δ', δ'', δ''' and where Δ' denotes the derivative of Δ. Define for $\varepsilon^{-1} r \in \mathbb{Z}$ and $\tau \geqslant 0$

$$\Delta_\varepsilon(r, \tau) = \delta_\varepsilon(\varepsilon^{-1} r, \varepsilon^{-2} \tau) \tag{5}$$

and let $\Delta_\varepsilon(r, \tau)$ for $r \in \mathbb{R}$ and $\tau \geqslant 0$ be its linear interpolation. Then

$$\lim_{\varepsilon \to 0} \Delta_\varepsilon(r, \tau) = \Delta(r, \tau) \tag{6}$$

where $\Delta(r, \tau)$ is in $C^\infty(\mathbb{R} \times \mathbb{R}_+)$ and it satisfies the equation

$$\frac{\partial}{\partial \tau} \Delta(r, \tau) = \frac{\partial}{\partial r} \left[U''(\Delta(r, \tau)) \frac{\partial}{\partial r} \Delta(r, \tau) \right] \tag{7a}$$

$$\Delta(r, 0) = \Delta(r) \tag{7b}$$

The proof of Theorem 1 is obtained in two steps: first we prove that $\Delta_\varepsilon(r, \tau)$ is an equicontinuous and bounded family of functions, so that by the Ascoli–Arzelà theorem[4] it converges by subsequences on the compacts. In the second step we prove that the limiting points satisfy Eq. (7), hence they coincide.

Equicontinuity. The function $U'(\delta)$: $\mathbb{R}_+ \to \mathbb{R}_+$ is invertible so we can change variables and go from $\delta(n, t)$ to $u(n, t)$, where

$$u(n, t) = U'(\delta(n, t)) \tag{8}$$

In the following one should think of $\delta(n, t)$ as a function of $u(n, t)$ via Eq. (8). From Eq. (1a) we get

$$\frac{\partial}{\partial t} u(n, t) = G(n, t)[u(n+1, t) + u(n-1, t) - 2u(n, t)] \tag{9a}$$

$$u(n, 0) = u(n) \tag{9b}$$

$$G(n, t) = U''(\delta(n, t)) \tag{9c}$$

Because of $G(n, t)$ Eq. (9a) is nonlinear. For the moment it is convenient to think of $G(n, t)$ as a *known* function. Then Eq. (9a) becomes a simple linear diffusive equation with a nice probabilistic interpretation. Fix $n \in \mathbb{Z}$ and $T > 0$. Let $x(t)$, $0 \leqslant t \leqslant T$, be a symmetric random walk on $\mathbb{Z}$ which starts from n and jumps on the nearest-neighbor sites with equal probability and intensity which depends on the time and the site where the walker is. The intensity g is given by

$$g(n, t) = G(n, T-t), \qquad n \in \mathbb{Z}, 0 \leqslant t \leqslant T \tag{10}$$

cf. Eq. (9c). Notice that by Eq. (2) g is uniformly bounded away from zero and infinity. We denote by $P_{n,T}$ the law of such walk and by $E_{n,T}$ its expectation. Then we obviously have

$$u(n, t) = E_{n,T}(u(x(T))) \tag{11}$$

where $u(n)$ is defined in Eq. (9b).

For the proof of equicontinuity the crucial estimate is the following:

Lemma 2. Let $\|u\|$ be the sup norm for the initial datum, i.e., $\|u\| \geqslant |u(n)|$ for all $n \in \mathbb{Z}$. Let a be as in Eq. (2). Then for every $T > 0$, n and n' in $\mathbb{Z}$,

$$|E_{n,T}(u(x(T))) - E_{n',T}(u(x(T)))| \leqslant c(aT)^{-1/2} |n - n'| \, \|u\| \tag{12}$$

where c is some "universal" constant.

Proof. Given n, n', and T we consider the following coupling Q between the random walks $x_1(t)$ and $x_2(t)$ which start from n and n' with marginal laws $P_{n,T}$ and $P_{n',T}$. The two walks, $x_1(t)$ and $x_2(t)$, move independently until they meet, at which time they stick together and from then on they move the same. Denoting by Q the law of such process and by E_Q its expectation, we have

$$\begin{aligned} |E_{n,T}(u(x(T))) - E_{n',T}(u(x(T)))| &= |E_Q(u(x_1(t)) - u(x_2(t)))| \\ &\leqslant \|u\| \, Q(\{x_1(T) \neq x_2(T)\}) \end{aligned} \tag{13}$$

The right-hand side of Eq. (13) is estimated in terms of the probability that two random walks which move independently do not meet before T and this is then reduced to the corresponding probability for two independent walks which move with constant intensity a; cf. Eq. (2). To do that we argue as follows. We consider the difference process $x_1(t) - x_2(t)$, $0 \leqslant t \leqslant T$. While the times of jump for such process depend on the past history of $x_1(t)$ and $x_2(t)$ this is not so for the value of the jump itself, which is always ± 1 with probability 1/2. Hence if we look at the discrete time process of the jumps of $x_1(t) - x_2(t)$ this is a simple symmetric random walk. The law of the jumping times in the Q process is stochastically smaller than the law of the jumping times for independent walks each with constant intensity a, because the jumps in the Q process have intensity larger than $2a$; cf. Eqs. (9c) and (2).

The proof of Eq. (12) is then completed by using classical estimates on the return time to the origin for a simple random walk. ∎

Remarks to Lemma 2. From Lemma 2 it [easily] follows that if $\delta(n, t) = \delta(n, 0) > 0$ for all n and some $t > 0$, then there exists δ such that $\delta(n, t) = \delta$ for all n and t. Hence all the stationary profiles are constant profiles.

We pose

$$v_\varepsilon(r, \tau) = U'(\Delta_\varepsilon(r, \tau)) \tag{14}$$

and we will first prove equicontinuity for the vs. For $\varepsilon^{-1} r \in \mathbb{Z}$

$$v_\varepsilon(r, \tau) = u^\varepsilon(\varepsilon^{-1} r, \varepsilon^{-2} \tau) \tag{15}$$

where u^ε is the solution of Eq. (9) with initial value

$$u^\varepsilon(n) = U'(\Delta(\varepsilon n)) \tag{16}$$

Hence for $\varepsilon^{-1} r \in \mathbb{Z}$ and $\tau \geqslant 0$

$$v_\varepsilon(r, \tau) = E_{\varepsilon^{-1} r}[U'(\Delta(\varepsilon x(\varepsilon^{-2}\tau)))] \tag{17}$$

We need to show that given any (r, τ) and $\zeta > 0$ there are R and T positive so that the following holds. For any r' and τ' such that $|r - r'| \leqslant R$ and $|\tau - \tau'| \leqslant T$, $\tau' \geqslant 0$,

$$|v_\varepsilon(r, \tau) - v_\varepsilon(r', \tau')| < \zeta \tag{18}$$

The above easily follows from the following considerations:

(1) By Lemma 2 for $\tau > 0$ fixed,

$$|u^\varepsilon(n, \varepsilon^{-2}\tau) - u^\varepsilon(n', \varepsilon^{-2}\tau)| \leqslant c(a\tau)^{-1/2} \, |\varepsilon n - \varepsilon n'| \tag{19}$$

hence by Eq. (15) the equal time estimate in Eq. (18) follows.

(2) By Eq. (11) we can write for $t' > t$

$$u(n, t') = E_{n,t'}(u(x(t'-t), t) \tag{20}$$

We can reduce then Eq. (18) to an estimate at equal times. We need to control

$$|x(t'-t) - n|$$

We notice that there exist constant c' and c'' so that for any $\lambda \geqslant 0$

$$P_{n,T}(\{\sup_{0 \leqslant t \leqslant T} |x(t) - n| \geqslant \lambda (bT)^{1/2}\}) \leqslant c' \exp(-c''\lambda) \tag{21}$$

where b is the constant appearing in Eq. (2). In fact the jumps of $x(t)$ are always ± 1 with probability 1/2 and the times of the jumps are stochastically larger than those of a random walk which moves with constant intensity b. Equation (21) becomes then a classical estimate for simple symmetric random walks.

(3) It remains to consider the case when we are given (r, τ) with $\tau = 0$, because in such case we cannot use Lemma 2. It follows from point (2) above that $|x(\varepsilon^{-2}\tau) - \varepsilon^{-1}r|$ is of order $\varepsilon^{-1}\sqrt{\tau}$. By Eq. (17) and the assumed smoothness of U' and Δ we then get that

$$|v_\varepsilon(r, \tau) - v_\varepsilon(r)| < \zeta$$

for $\tau < T_\zeta$ and T_ζ can be chosen independently of ε and r.

So far we have proven that $v_\varepsilon(r, \tau)$ converges by subsequences, then the same holds also for $\Delta_\varepsilon(r, \tau)$. Let $v(r, \tau)$ and $\Delta(r, \tau)$ be some limiting values for v_ε and Δ_ε, respectively, then $v(r, \tau) = U'(\Delta(r, \tau))$ Furthermore from Eq. (17) and taking the limit as ε goes to zero we obtain that

$$v(r, \tau) = E^0_{r,\tau}(v(x(\tau)) \tag{22}$$

$$v(r) = U'(\Delta(r)) \tag{23}$$

where for fixed $\tau > 0$ and $0 \leqslant t \leqslant \tau$, $r \in \mathbb{R}$, $P^t_{r,\tau}$ denotes the law of the process $x(t')$, $t \leqslant t' \leqslant \tau$, which starts at r at time t and satisfies the equation

$$dx(t') = \sigma(x(t'), t')\, db(t') \tag{24}$$

where $b(t)$ is a standard Brownian motion and

$$\sigma(r, t') = G(r, \tau - t')^{1/2}, \qquad G(r, t) = U''(\Delta(r, t)) \tag{25}$$

$E^t_{r,\tau}$ denotes the corresponding expectation. We shall prove that there is only one continuous bounded function v for which Eqs. (22) and (23) hold [with the conditions Eqs. (24) and (25)] and this will complete the proof of Theorem 1. The argument is a straightforward adaptation to our context of Theorem 2.1 of Ref. 1; cf. also Ref. 2. We start noticing that there exists a C^∞ function $v(r, t)$ for which Eqs. (22)–(25) hold, this is the "classical" solution of

$$\frac{\partial}{\partial\tau} v(r, \tau) = G(r, \tau) \frac{\partial^2}{\partial r^2} v(r, \tau), \qquad v(r, 0) = v(r) \tag{26}$$

Let us now assume that there is some other function $\bar{v}(r, \tau)$ such that

$$\bar{v}(r, \tau) = E^0_{r,\tau}(v(y(\tau))) \tag{27}$$

$$dy(t') = \bar{\sigma}(y(t'), t')\, db(t') \tag{28}$$

We fix $\tau > 0$ and for $0 \leqslant t \leqslant s \leqslant \tau$ we set

$$\gamma^t(s) = \sup_r E^t_{r,\tau} |x(s) - y(s)|^2 \tag{29}$$

$$\Delta(s) = \sup_r |v(r, s) - \bar{v}(r, s)|^2 \tag{30}$$

By using Eqs. (28) and (29) we have for $0 \leqslant t \leqslant s \leqslant \tau$

$$\begin{aligned} E^t_{r,\tau} |x(s) - y(s)|^2 &\leqslant \int_t^s dt'\, E^t_{r,\tau} |\sigma(x(t'), t') - \bar{\sigma}(y(t'), t')|^2 \\ &\leqslant 2 \int_t^s dt'\, E^t_{r,\tau}(|\sigma(x(t'), t') - \sigma(y(t'), t')|^2 \\ &\quad + |\sigma(y(t'), t') - \bar{\sigma}(y(t'), t')|^2) \\ &\leqslant 2 \int_t^s dt' [\|\sigma'\|^2 \gamma^t(t') + \Delta(t')] \end{aligned}$$

hence

$$\gamma^t(s) \leqslant 2 \int_t^s dt' [\|\sigma'\|^2 \gamma^t(t') + \Delta(t')] \tag{31}$$

$$\Delta(t) \leqslant \|v'\|^2 \gamma^t(\tau) \tag{32}$$

where $\|\sigma'\|$ and $\|v'\|$ are the sup norm for $\sigma'(r, t)$ and $v'(r)$. From Eqs. (31) and (32) uniqueness follows for small τ. Iterating the argument uniqueness is proven for all times.

Concluding Remarks. (1) One does not need to use PDE theory to have a classical solution of Eq. (26) since one can prove directly, like in

Ref. 1, that there is a Lipshitz continuous solution of the problem stated in Eqs. (22)–(25): this is enough for deriving Eqs. (31) and (32).

(2) For uniqueness we only need to know that there is a solution which is Lipshitz continuous: from here uniqueness follows in the larger class of *only* continuous functions.

(3) One of the difficulties in Ref. 5 comes from the fact that at the last step of the proof one needs to know that the limiting function is Lipshitz continuous. We avoid that by using the method outlined above. We remark that we do not have Lipshitz continuity of $v_\varepsilon(r, \tau)$ by using Lemma 2, because of the factor $(\tau)^{-1/2}$ in Eq. (16).

(4) In the interpretation of Eq. (1) as a model of traffic, $\delta(n)$ represents the distance between car $n+1$ and car n. Equation (1) is related then to the set of equations

$$\frac{\partial}{\partial t}\omega(n, t) = U'(\delta(n, t)) - U'(\delta(n-1, t)) \tag{33a}$$

$$\omega(n+1, t) - \omega(n, t) = \delta(n, t) \tag{34}$$

Our previous considerations extend to this case since we have

$$\frac{\partial}{\partial t}\omega(n, t) = \Omega(n, t)[\omega(n+1, t) + \omega(n-1, t) - 2\omega(n, t)] \tag{35}$$

where $\Omega(n, t)$ is the first order Taylor Lagrange expension for the right-hand side of Eq. (33a).

ACKNOWLEDGMENTS

This research was partially supported by NATO grant No. 040.82.

We thank J. Fritz, G. Benfatto, and M. Pulvirenti for many helpful discussions and comments.

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